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A Lagrangian of the quasi-rigid extended charge

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Abstract
A Lagrangian is proposed for the quasi-rigid extended charged particle, which consists of a bare point particle term plus the standard electromagnetic minimal coupling. The quasi-rigid motion is imposed as a constraint. The extension of the particle and the quasi-rigid motion appear inside the current density. The Lorentz contraction of the extended particle makes the interaction term dependent on the acceleration. This dependence produces the additional terms in the equations of motion that are necessary for the proper energy and momentum conservation, and that were previously identified as the inertial effects of stress. The momentum of stress is obtained as an explicit function of the electromagnetic field.

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1. Introduction

In a previous article [1] the electrodynamics of a classical extended charge was studied from various points of view. It was assumed that the particle follows a quasi-rigid motion. The main results of that paper are the following.

(1) The $4/3$ problem is solved. As it is well known that the momentum and energy of the electromagnetic fields that surround a spherically symmetric charge distribution moving with velocity $\boldsymbol{v}$ are respectively $\frac{4}{3}U_\mathrm{e}\gamma\boldsymbol{v}/c^2$ and $U_\mathrm{e}\gamma\left[1+\frac{1}{3}(v/c)^2\right]$, were U_e is the electrostatic energy and $\gamma = [1-(v/c)^2]^{-1/2}$. These values of energy and momentum do not form a 4-vector and seem to contradict the mass–energy equivalence. One should expect that the mass of the dressed particle be the bare mass m_0 plus the electromagnetic mass U_e/c^2. It was shown that everything fits in place, once one considers the inertial effects of the stress that develops inside the particle to balance the electrostatic repulsion. For a particle moving with no acceleration those effects can be included in a negative pressure contribution to the mass m_P,

$$m_P = -\frac{1}{3c^2}U_\mathrm{e}. \tag{1}$$

So the mass of the dressed particle (bare + stress + bounded fields) has the expected value $m_0 + U_e/c^2$. The relevance of the inertial effects of stress is also discussed in [2]. In this respect it is worth mentioning the Boyer [3] and Rohrlich [4] controversy that was not discussed in the previous paper [1]. Boyer claimed that it was wrong to modify the electromagnetic energy–momentum tensor $T^{\mu\nu}$ in order to eliminate the extra $1/3$ term. Rohrlich maintained the opposite opinion. The controversy is settled in favour of Boyer. Rohrlich assumes, in his reply to Boyer, that the interaction that equilibrates the electrostatic repulsion enters in the equation as a force density, while actually it is the stress of the particle. The fact that the energy and the momentum of the fields do not form a 4-vector is due to the fact that not only the particle is stressed, but also the fields that surround it. $T^{\mu\nu}$ should be named more properly as the energy–momentum–stress tensor.

(2) It is established the role played by the various components of the $T^{\mu\nu}$ in the energy and momentum conservation. The fields produced by the particle have two components: the radiated field that decays as $1/r$ and the bound field that decays as $1/r^2$. As the tensor $T^{\mu\nu}$ is quadratic in the field there are three components T_{BB}, T_{BR} and T_{RR}. The radiation term T_{RR} gives the energy and momentum of the radiated fields. The term T_{BB} corresponds to the field bounded to the particle and gives the electromagnetic contributions to the dressed particle. The cross term T_{BR} is also bounded to the particle but it only exists as long as there is radiation. The radiation reaction contains a term that corresponds to radiated momentum, but also a term that corresponds to the cross term. This cross term behaves as reservoir of energy and momentum.

This splitting of $T^{\mu\nu}$ was found long ago by Teitelboim [5]. He obtained the Lorentz–Abraham–Dirac (LAD) radiation reaction formula for a point charge using the retarded fields. In the cited paper Teitelboim used the energy–momentum conservation law of the field. The energy–momentum contribution to the dressed particle was calculated in the reference frame of instantaneous rest of the particle, and then transformed into the laboratory frame assuming that energy and momentum form a 4-vector. In this way the stress contribution is suppressed. As everybody else he disregarded the stress contribution to the mass of the particle. As a result the expected mass of the dressed particle was obtained $(m_0 + e^2/(2\epsilon c^2))$. In a later paper [6] determined the radiation reaction by calculating directly the force produced by the self-fields. He obtained the LAD formula, but now, of course, the electromagnetic mass included the stress contribution and was $2e^2/(3\epsilon c^2)$. About this discrepancy he wrote in a note: 'However, there is actually no difference between the two expressions, since the limit $\epsilon \to 0$ is to be taken'. One cannot agree with that. Actually, the difference shows that the stress of the field makes a real contribution to the momentum of the particle.

(3) It is found an exact formula for the radiation reaction of the extended particle. The self-force is given as an integral over the retarded accelerations. In the $R \to 0$ limit the LAD result is recovered.

(4) It is found that the solutions of the integro-differential equation of motion that results when the exact radiation reaction formula is used do not violate causality or run away, provided that the mass of matter $m_0 + m_P$ is positive. When the condition

$$m_0 > \frac{U_e}{3c^2} \tag{2}$$

is not verified the causality is violated and run-away solutions appear. Therefore the point particle is inconsistent in classical electrodynamics, as $\lim_{R\to 0} U_e = +\infty$. That is, the limit $R \to 0$ is not physical as it is interior to a non-physical region. The classical mass renormalization is also inconsistent. It is impossible to verify (2) and to go to $-\infty$ at the same time.

(5) It is shown that for a physical point particle that verifies (2) the exact radiation reaction formula reduces to the Rohrlich formula [7]. A physical point particle is a particle whose radius is much smaller than any other distance in the problem, in particular than the wave-length of the fields that it itself generates. The Rohrlich formula is like the LAD formula but replacing the acceleration $\boldsymbol{a}$ by the external force $\boldsymbol{F}$ divided by the dressed mass m.

(6) Finally it is shown that the radiated power of the physical point charge is not given by the Larmor formula, which is valid for $R \to 0$, but by a modified one.

The conclusion of the previous paper [1] is that the dynamics of a classical quasi-rigid extended charge, including self-interactions, is, unlike that of a point charge, perfectly consistent and conforming with causality and conservation of energy and momentum.

The extended particle should have some kind of structure that generates the stress that balances the electrostatic repulsion. Nevertheless if a particle of radius R moves with an acceleration which is small in comparison with c^2/R, it continuously keeps its spherical shape as seen in the reference frame of instantaneous rest. In this quasi-rigid motion the internal dynamics is frozen, so the particle moves as its only three degrees of freedom were the coordinates of its centre. The quasi-rigid motion corresponds to a constraint that eliminates the internal degrees of freedom. It has been shown [8] that the pressure mass is the same for any elasticity model with spherical symmetry, so the elastic properties should not be relevant in the quasi-rigid limit.

Here we show that the mechanics of the quasi-rigid extended particle can be obtained from the standard electromagnetic Lagrangian with the minimal coupling. The peculiarity of the extended particle is that the current density depends on acceleration. This is due to the fact that as the speed changes the Lorentz contraction changes and that, therefore, different points of the particle should move with different velocities. Such dependence on the acceleration is unavoidable. Higher order Lagrangians are rare. One may imagine elasticity models of the particle that correspond to first order Lagrangians. The interaction which is proportional to the velocity of each point will also be of first order. It is the quasi-rigid constraint, that makes the motion of each point of the particle a function of the motion of its centre, what produces the acceleration dependent interaction. That is the price one has to pay for having eliminated the internal degrees of freedom.

2. The Lagrangian

We will assume that only electromagnetic forces are acting on the particle, but that, in addition to the fields generated by the particle itself, there are also those due to some external current density j^μ_{ex}. We will use Gauss electromagnetic units and the metric tensor $g^{\mu\nu}$ with positive trace. We will call z^μ a generic point of four-space and x^μ the coordinates of the centre of the particle, both in the laboratory frame. The origin of the instantaneous rest frame will be the centre of the particle while y^μ will be the generic point and $y = |\boldsymbol{y}|$. We will assume that the particle has a non-zero charge q and that in the instantaneous rest frame it has a constant relative charge density $g(y)$, which has spherical symmetry and is normalized to 1

$$\int \mathrm{d}^3y\, g(y) = 1. \tag{3}$$

Each point of the particle can be labelled with its position in the instantaneous rest frame, y^μ. The quasi-rigid motion is defined in [1] so that the position of any point of the particle is given at any time by

$$\boldsymbol{x}(\boldsymbol{y}, t) = \boldsymbol{x}(t) + \boldsymbol{y} + \delta\boldsymbol{y}, \tag{4}$$

where $\delta\boldsymbol{y}$ is the Lorentz contraction

$$\delta\boldsymbol{y} = (\gamma^{-1} - 1)(\boldsymbol{y} \cdot \hat{\boldsymbol{v}})\hat{\boldsymbol{v}}. \tag{5}$$

The quantity γ is calculated with the velocity of the centre $\boldsymbol{v}$ and $\hat{\boldsymbol{v}}$ is the unit vector in the direction of $\boldsymbol{v}$. The actual motion of a point of the particle differs from expression (4) by a term of order yaR/c^2, where a is the acceleration.

Expression (4) can be inverted,

$$\boldsymbol{y} = \boldsymbol{z} - \boldsymbol{x} + (\gamma - 1)(\boldsymbol{z} - \boldsymbol{x}) \cdot \hat{\boldsymbol{v}}\hat{\boldsymbol{v}}. \tag{6}$$

Because $\delta\boldsymbol{y}$ depends on $\boldsymbol{v}$, different points of the particle have different velocities as the particle is accelerated.

$$\boldsymbol{v}(\boldsymbol{y}, t) = \frac{\partial \boldsymbol{x}(\boldsymbol{y}, t)}{\partial t} \tag{7}$$

$$= \boldsymbol{v}(t) + \delta\boldsymbol{v}(\boldsymbol{y}, t), \tag{8}$$

where

$$\delta v_i = \frac{\partial \delta y_i}{\partial t} \tag{9}$$

$$= a_j \frac{\partial \delta y_i}{\partial v_j}. \tag{10}$$

In the instantaneous rest frame the charge density is $\rho = qg(y)$ and the current density vanishes. In the laboratory frame the charge density is

$$\rho(z^\mu) = q\gamma g(|\boldsymbol{z} - \boldsymbol{x} + (\gamma - 1)(\boldsymbol{z} - \boldsymbol{x}) \cdot \hat{\boldsymbol{v}}\hat{\boldsymbol{v}}|) \tag{11}$$

$$= q \int \mathrm{d}^3 y\, g(y)\delta(\boldsymbol{z} - \boldsymbol{x} - \boldsymbol{y} - \delta\boldsymbol{y}) \tag{12}$$

and the current density is

$$\boldsymbol{j}(z^\mu) = \rho(z^\mu)(\boldsymbol{v} + \delta\boldsymbol{v}) \tag{13}$$

$$= q \int \mathrm{d}^3 y\, g(y)\delta(\boldsymbol{z} - \boldsymbol{x} - \boldsymbol{y} - \delta\boldsymbol{y})(\boldsymbol{v} + \delta\boldsymbol{v}). \tag{14}$$

In both expressions we have used the fact that $\mathrm{d}^3 y = \gamma\, \mathrm{d}^3 z$. Definitions (11) and (13) are consistent with the charge conservation $\partial_\mu j^\mu = 0$.

We can now write down the Lagrangian we propose, namely

$$L(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}, t, A^\mu) = -\frac{1}{16\pi} \int \mathrm{d}^3 z\, F^{\mu\nu} F_{\mu\nu} - m_0 c^2 \gamma^{-1} + \frac{1}{c} \int \mathrm{d}^3 z \big(j^\mu + j^\mu_{\mathrm{ex}}\big) A_\mu. \tag{15}$$

The first term is the Lagrangian of electromagnetic fields, the second is the Lagrangian of a bare point particle and the third is the standard electromagnetic coupling. The fact that the particle has extension and that its motion is quasi-rigid appears in the current density j^μ. The dependence on the acceleration $\boldsymbol{a}$ is in $\delta\boldsymbol{v}$. Using the expressions for the current density, the interaction term of the Lagrangian can be written as

$$L_{\mathrm{I}} = \frac{q}{c} \int \mathrm{d}^3 y\, g(y)(\boldsymbol{v} + \delta\boldsymbol{v}) \cdot \boldsymbol{A}(\boldsymbol{x} + \boldsymbol{y} + \delta\boldsymbol{y}, t) - q \int \mathrm{d}^3 y\, g(y)\phi(\boldsymbol{x} + \boldsymbol{y} + \delta\boldsymbol{y}, t). \tag{16}$$

It is obvious that this Lagrangian yields the correct Maxwell equations. We will show that it also gives the correct equations of motion of the particle, but before we will in the next section recall how to handle Lagrangians that depend on acceleration.

3. Acceleration dependent Lagrangian

The treatment of higher order Lagrangians was developed by Ostrogradsky in the middle of 19th century [9]. The Hamiltonian approach was made by Govaerts and Rashid [10]. We resume here the results for an acceleration dependent Lagrangian $L(q, \dot{q}, \ddot{q}, t)$. The conjugate momentum of q_i is

$$p_i = \frac{\partial L}{\partial \dot{q}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}_i}. \tag{17}$$

The Euler–Lagrange equations of motion are

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \frac{\partial L}{\partial q_i}. \tag{18}$$

The quantity $\frac{\partial L}{\partial \ddot{q}_i}$ behaves as the conjugate momentum of $\dot{q}_i$, so the Hamiltonian is

$$H = \sum_i \dot{q}_i p_i + \sum_i \ddot{q}_i \frac{\partial L}{\partial \ddot{q}_i} - L. \tag{19}$$

Finally the evolution of H is given by

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial L}{\partial t}. \tag{20}$$

4. Equations of motion

In this section we obtain the equations of motion of the Lagrangian (15). The conjugate momentum of $\boldsymbol{x}$ is

$$p_i = \frac{\partial L}{\partial v_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial a_i}. \tag{21}$$

From (16) and (10) we obtain

$$\frac{\partial L}{\partial a_i} = \frac{q}{c}\int \mathrm{d}^3y\, g(y)\frac{\partial \delta v_j}{\partial a_i}A_j \tag{22}$$

$$= \frac{q}{c}\int \mathrm{d}^3y\, g(y)\frac{\partial \delta y_j}{\partial v_i}A_j, \tag{23}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial a_i} = \frac{q}{c}\int \mathrm{d}^3y\, g(y)\frac{\partial \delta y_j}{\partial v_i}\left[\frac{\partial A_j}{\partial t} + (v_k + \delta v_k)\frac{\partial A_j}{\partial x_k}\right] + \frac{q}{c}\int \mathrm{d}^3y\, g(y)a_k\frac{\partial^2 \delta y_j}{\partial v_k \partial v_i}A_j. \tag{24}$$

On the other hand from (15), (16) and (10) we get

$$\frac{\partial L}{\partial v_i} = m_0\gamma v_i + \frac{q}{c}\int \mathrm{d}^3y\, g(y)A_i + \frac{q}{c}\int \mathrm{d}^3y\, g(y)\frac{\partial \delta v_j}{\partial v_i}A_j$$
$$+ \frac{q}{c}\int \mathrm{d}^3y\, g(y)\frac{\partial \delta y_j}{\partial v_i}\left[(v_k + \delta v_k)\frac{\partial A_k}{\partial x_j} - c\frac{\partial \phi}{\partial x_j}\right]. \tag{25}$$

The conjugate momentum is then

$$p_i = m_0\gamma v_i + \frac{q}{c}\int \mathrm{d}^3y\, g(y)A_i + q\int \mathrm{d}^3y\, g(y)\frac{\partial \delta y_j}{\partial v_i}[\boldsymbol{E} + c^{-1}(\boldsymbol{v} + \delta\boldsymbol{v}) \times \boldsymbol{B}]_j, \tag{26}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electrical and magnetic fields respectively.

The first term of (26) is the momentum of the bare particle, the second term is the usual vector potential contribution which is also present for the point particle, but in this case it is averaged over the whole particle. The last term is distinctive of the extended quasi-rigid particle. So, we will define the momentum of constraint as

$$p_{Ci} = q \int \mathrm{d}^3 y\, g(y) \frac{\partial \delta y_j}{\partial v_i} [\boldsymbol{E} + c^{-1}(\boldsymbol{v} + \delta \boldsymbol{v}) \times \boldsymbol{B}]_j. \tag{27}$$

The largest contribution to the constraint momentum comes from the electrostatic repulsion of the charges of the particle. We will call momentum of matter $\boldsymbol{p}_M$ the sum of the bare momentum plus the constraint momentum

$$\boldsymbol{p}_M = m_0 \gamma \boldsymbol{v} + \boldsymbol{p}_C. \tag{28}$$

With these definitions the conjugate momentum is

$$\boldsymbol{p} = \boldsymbol{p}_M + \frac{q}{c} \int \mathrm{d}^3 y\, g(y) \boldsymbol{A}. \tag{29}$$

The equation of motion of $\boldsymbol{p}_M$ is obtained from (18)

$$\dot{\boldsymbol{p}}_M = \dot{\boldsymbol{p}} - \frac{q}{c}\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3 y\, g(y) \boldsymbol{A} \tag{30}$$

$$= \frac{\partial L}{\partial \boldsymbol{x}} - \frac{q}{c} \int \mathrm{d}^3 y\, g(y) \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{A} \tag{31}$$

$$= \frac{q}{c} \int \mathrm{d}^3 y g(y) \left[\nabla(\boldsymbol{v} + \delta \boldsymbol{v}) \cdot \boldsymbol{A} - c \nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t} - (\boldsymbol{v} + \delta \boldsymbol{v}) \cdot \nabla \boldsymbol{A} \right] \tag{32}$$

$$= q \int \mathrm{d}^3 y\, g(y) [\boldsymbol{E} + c^{-1}(\boldsymbol{v} + \delta \boldsymbol{v}) \times \boldsymbol{B}] \tag{33}$$

$$= \int \mathrm{d}^3 z (\rho \boldsymbol{E} + c^{-1} \boldsymbol{j} \times \boldsymbol{B}). \tag{34}$$

To call $\boldsymbol{p}_M$ the momentum of matter is justified by the fact that its time derivative is the integral of the force density. Note that it is different from the dressed particle momentum, which in addition includes the momentum of the fields that surround the particle. The equation of motion (34) is similar to equation (14) of [1], but there instead of $\boldsymbol{p}_C$ one has the momentum of stress $m_P \gamma \boldsymbol{v}$. In [1, 8] the mass of stress m_P is calculated from the stress tensor of the particle, and the stress is determined from the stability condition of the particle. Instead expression (27) gives the momentum of stress as an explicit function of the fields.

5. Energy equation

The energy is obtained from (19),

$$E = \boldsymbol{v} \cdot \boldsymbol{p} + \boldsymbol{a} \cdot \frac{\partial L}{\partial \boldsymbol{a}} - L \tag{35}$$

$$= m_0 c^2 \gamma + \boldsymbol{v} \cdot \boldsymbol{p}_C + q \int \mathrm{d}^3 y\, g(y) \phi. \tag{36}$$

The first term is the bare particle energy, the last one is the electrostatic potential energy and the second term is a contribution due to the constraint. As for the momentum we define the energy of matter as the sum of the bare particle contribution plus the constraint contribution

$$E_M = m_0 c^2 \gamma + \boldsymbol{v} \cdot \boldsymbol{p}_C. \tag{37}$$

Note that E_M and $\boldsymbol{p}_M$ do not form a 4-vector.

The time evolution of energy is obtained using (20)

$$\frac{\mathrm{d}E_M}{\mathrm{d}t} = -\frac{\partial L}{\partial t} - q\int \mathrm{d}^3y\, g(y)\frac{\partial \phi}{\partial t} \tag{38}$$

$$= q\int \mathrm{d}^3y\, g(y)\left[-c^{-1}(\boldsymbol{v}+\delta\boldsymbol{v})\cdot\frac{\partial \boldsymbol{A}}{\partial t} + \frac{\partial \phi}{\partial t} - \frac{\mathrm{d}\phi}{\mathrm{d}t}\right] \tag{39}$$

$$= -q\int \mathrm{d}^3y\, g(y)(\boldsymbol{v}+\delta\boldsymbol{v})\cdot\left(c^{-1}\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi\right) \tag{40}$$

$$= q\int \mathrm{d}^3y\, g(y)(\boldsymbol{v}+\delta\boldsymbol{v})\cdot\boldsymbol{E} \tag{41}$$

$$= \int \mathrm{d}^3z\, \boldsymbol{j}\cdot\boldsymbol{E}. \tag{42}$$

The time derivative of E_M is the integral of power density. This last equation corresponds to equation (17) of [1]. The pressure contribution to the energy that appears there can be written as $m_P\gamma v^2$, which is consistent with the constraint term of (36).

6. Identity of the stress momentum and the constraint momentum

In this section we will calculate the constraint momentum with the same approximations that were used in [1], that is: (1) the size of the particle is small in comparison with the external currents, so the external fields can be considered constant inside the particle; (2) the dependence on acceleration will be neglected. With these conditions the only contributions to the constraint momentum in (27) come from the electrostatic self-field. In the rest frame the magnetic self-field vanishes, while the electric field is

$$\boldsymbol{E} = q\frac{Q(y)}{y^2}\hat{\boldsymbol{y}} \tag{43}$$

where $Q(y)$ is

$$Q(y) = \int_0^y \mathrm{d}y\, 4\pi y^2 g(y). \tag{44}$$

In the laboratory frame, expressed in terms of the coordinate of the rest frame $\boldsymbol{y}$, the electric and magnetic self-fields are

$$\boldsymbol{E} = q\frac{Q(y)}{y^2}[\gamma\hat{\boldsymbol{y}} + (1-\gamma)(\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{v}})\hat{\boldsymbol{v}}] \tag{45}$$

and

$$\boldsymbol{B} = \frac{\gamma q}{cy^2}Q(y)\boldsymbol{v}\times\hat{\boldsymbol{y}}. \tag{46}$$

As $\boldsymbol{a} = 0$ then $\delta\boldsymbol{v} = 0$, so the bracket in (27) is

$$\boldsymbol{E} + c^{-1}\boldsymbol{v}\times\boldsymbol{B} = q\frac{Q(y)}{y^2}[\gamma^{-1}\hat{\boldsymbol{y}} + (1-\gamma^{-1})(\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{v}})\hat{\boldsymbol{v}}]. \tag{47}$$

On the other hand

$$\frac{\partial\delta y_j}{\partial v_i} = -\frac{yv}{c^2(\gamma^{-1}+1)}[(\gamma-1)(\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{v}})\hat{v}_i\hat{v}_j + \hat{y}_i\hat{v}_j + (\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{v}})\delta_{ij}]. \tag{48}$$

Therefore, from (47) and (48)

$$\frac{\partial \delta y_j}{\partial v_i}[\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{B}]_j = -\frac{q Q(y)}{c^2 y}[(\gamma - 1)(\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{v}})^2 v_i + (\boldsymbol{v} \cdot \hat{\boldsymbol{y}})\hat{y}_i]. \quad (49)$$

As the particle has spherical symmetry, in doing the integral of (27) one can first integrate the solid angle, and then the radial coordinate. The spherical average of (49) is readily obtained using the fact that

$$\frac{1}{4\pi}\int \mathrm{d}\Omega \hat{y}_i \hat{y}_j = \frac{1}{3}\delta_{ij}. \quad (50)$$

The average is

$$\frac{1}{4\pi}\int \mathrm{d}\Omega \frac{\partial \delta y_j}{\partial v_i}[\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{B}]_j = -\frac{q Q(y)\gamma}{3c^2 y} v_i \quad (51)$$

and therefore the constraint momentum is

$$\boldsymbol{p}_C = -\frac{q^2\gamma}{3c^2}\int \mathrm{d}^3 y \frac{g(y)Q(y)}{y}\boldsymbol{v}. \quad (52)$$

The integral in (52) is proportional to the electrostatic energy

$$\int \mathrm{d}^3 y \frac{g(y)Q(y)}{y} = \int \mathrm{d}Q \frac{Q(y)}{y} \quad (53)$$

$$= \frac{Q(y)^2}{2y}\bigg|_0^\infty + \frac{1}{2}\int \mathrm{d}y \frac{Q(y)^2}{y^2} \quad (54)$$

$$= \frac{1}{8\pi q^2}\int \mathrm{d}^3 y E^2 \quad (55)$$

$$= q^{-2} U_e. \quad (56)$$

The constraint momentum is then

$$\boldsymbol{p}_C = m_P \gamma \boldsymbol{v} \quad (57)$$

where m_P is given in (1). It is exactly the same expression of the momentum of stress that was given in [1, 8]. The energy of matter E_M is also the same that appears in equation (17) of [1],

$$E_M = m_0 c^2 \gamma + m_P \gamma v^2 \quad (58)$$

$$= (m_0 + m_P)c^2\gamma - m_P c^2 \gamma^{-1}. \quad (59)$$

7. Conclusion

We have shown that the standard electromagnetic Lagrangian with minimal coupling $j^\mu A_\mu$ yields the proper behaviour of the quasi-rigid extended particle. The internal degrees of freedom are not included; instead the quasi-rigid motion is imposed as a constraint. The velocities of different parts of the particle are different when the particle is accelerated, so the current density $\boldsymbol{j}$ and the Lagrangian depend on acceleration. This fact produces additional terms in the momentum and energy of the particle that are the same that were found in the previous work [1] to be the inertial effects of stress. These additional terms exactly cancel the additional terms in the energy and momentum of the self-fields that surround the particle, and therefore the dressed particle (bare particle + constraint + surrounding fields) has a standard momentum–energy 4-vector corresponding to the expected mass $m_0 + U_e/c^2$. All the results of [1], in particular the correct radiation reaction formula, are consistent with the present Lagrangian formulation.

To have found a proper Lagrangian theory is the first step towards the quantization of the extended quasi-rigid particle, but the quantization of acceleration-dependent Lagrangians is not straightforward. A possible path that may be followed in order to achieve this goal could be to convert the Lagrangian to a first order one by considering the velocity $\boldsymbol{v}$ as a generalized coordinate independent from $\boldsymbol{x}$ and to impose the condition $\dot{\boldsymbol{x}} = \boldsymbol{v}$ as a constraint by means of Lagrange's multipliers. Such singular Lagrangian could be quantized using Dirac's method [10].

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